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Lipschitzian Selections in Best Approximation by Continuous Functions*

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The problem under consideration is to find a best uniform approximation to a function f from a set K in the space of continuous functions. Conditions are derived on K such that the selection operator mapping f to one of its best approximations is Lipschitzian. Their application is illustrated by approximation problems.

1. INTRODUCTION

The main thrust of this work is to derive a set of conditions which ensure existence of Lipschitzian selections in approximation problems involving continuous functions. Their application is illustrated with relevant examples in approximation. In an earlier paper [11], a set of conditions was developed by the author for similar investigations for the space of bounded functions. It was found that these conditions can be modified and applied uniformly to the spaces both of continuous functions and of bounded functions yielding results on selections. A partial converse indicating the necessity of some of these conditions is established. Their application to continuous functions, however, leads to special situations requiring further analysis which is also presented.

Let S be any set. Let $B = B(S)$ denote the space of real bounded functions on S with the uniform norm $\|\cdot\|$. Similarly, when S is topological, let $C = C(S)$ denote the space of real bounded and continuous functions on S, again, with the uniform norm. For convenience, let $X = B$ of C. Let $K \subset X$ be a nonempty set. Given f in X, let $\Delta(f)$ denote the infimum of $\|f-k\|$ for k in K. The problem is to find a best approximation f' in K such that

$$\Delta(f) = \|f - f'\| = \inf\{\|f-k\| : k \in K\}. \quad (1.1)$$

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A selection operator T is a nonlinear operator which maps each f in X to a best approximation f'. Such an operator T is a Lipschitzian selection operator (LSO) if it satisfies, for some least number $c(T)$,

$$\|T(f)-T(h)\| \leqslant c(T)\,\|f-h\|$$

for all f, h in X. An LSO T is an optimal Lipschitzian selection operator (OLSO) if $c(T) \leqslant c(T')$ for all LSO's T'. We develop conditions on K under which these operators can be identified.

For f in X, define $K_f=\{k \in K: k \leqslant f\}$ and $K'_f=\{k \in K: k \geqslant f\}$. Let

$$\bar{f}(s)=\sup\{k(s): k \in K_f\}, \qquad s \in S,$$

$$\underline{f}(s)=\inf\{k(s): k \in K'_f\}, \qquad s \in S.$$

The three conditions stated below are identical for $X=B$ or C.

(1) If $k \in K$, then $k+\alpha \in K$ for all real α.

(2) If $f \in X$, then $\bar{f} \in K$.

(3) If $f \in X$, then $\underline{f} \in K$.

When $\bar{f}$ and $\underline{f}$ are in K, they are called, respectively, the greatest K-minorant and the smallest K-majorant of f. Note that condition (2) ((3)) implies that the pointwise maximum (minimum) of two functions in K is also in K.

In Section 2, under conditions (1), (2) or (1), (3), we identify an LSO T with $c(T)=2$. If K is convex and all conditions hold, then we identify an OLSO T with $c(T)=1$. In each of these cases, T maps f to its maximal or minimal best approximation or their mean. We also consider another problem. Given f in X, let $\bar{\Delta}(f)$ denote the infimum of $\|f-k\|$ for k in K_f. The problem is to find an f' in K_f such that

$$\Delta(f)=\|f-f'\|=\inf\{\|f-k\|: k \in K_f\}. \tag{1.2}$$

Under conditions (1) and (2) on K we identify the unique OLSO T with $C(T)=1$ for this problem. In this case T maps f to its maximal best approximation. An LSO mapping f to its maximal or minimal best approximation is shown to be an extreme point of all selection operators. A partial converse is also provided which shows that the existence of a maximal best approximation for Problems (1.1) and (1.2) implies condition (2). We remark that all the results of Section 2 except possibly the exact values of $c(T)$ remain valid for any subspace $X' \subset B$ and $K \subset X'$ which satisfy the conditions stated.

Examples of approximation problems on continuous functions are given in Section 4. If S is a polytope, then conditions (1) and (2) hold for the problem of approximation by convex functions. Similarly, if S is compact and convex they apply to approximation by quasi-convex functions. The third example in Section 4 is approximation by isotone functions on a rectangle in R^n to which all conditions can be applied. Similarly, all conditions hold for the last problem of approximation by Lipschitzian functions. An analysis of the distance function specifying the distance of a point from a nonconvex set is presented in Section 3. The problems of L_p approximation by convex and quasi-convex functions are considered in [12]. Finally, we note that results on continuous and Lipschitzian continuous selections are provided in [2–4, 9, 10, 14].

2. Main Results

In this section we present results on Lipschitzian selections. Theorems 3.1 and 3.2 of [11] remain valid for $K \subset B$ or $K \subset C$ if conditions (i), (ii), (iii) of that article are replaced, respectively, by the weaker conditions (1), (2), (3) of Section 1. These two theorems are presented below in an abbreviated form and their proofs are outlined. Theorem 2.3, which is the partial converse mentioned in Section 1, is given later. An f' is a maximal (minimal) best approximation to f if $f' \geqslant g$ ($f' \leqslant g$) for all best approximations g to f. By a nonconvex set we mean a set which is not necessarily convex.

THEOREM 2.1. *The following applies to Problem* (1.1) *with $X = B$ or C and $K \subset X$.*

(a) *K nonvonvex. If K satisfies conditions* (1) *and* (2), *then* $\Delta(f) = \|f - \bar{f}\|/2$, *and* $f' = \bar{f} + \Delta(f)$ *is the maximal best approximation to f. Furthermore,* $\|f' - h'\| \leqslant 2\,\|f - h\|$ *for all* $f, h \in X$. *The operator* $T\colon X \to K$ *defined by* $T(f) = f'$ *is an* LSO *with* $c(T) = 2$.

(b) *K nonconvex. If K satisfies conditions* (1) *and* (3), *then* (a) *holds with* $\bar{f}$ *replaced by* $\underline{f}$ *and* $f' = \underline{f} - \Delta(f)$, *which is the minimal best approximation to f.*

(c) *K convex. If K satisfies conditions* (1), (2), *and* (3), *then* (a) *and* (b) *above apply. Also,* $\Delta(f) = \|\underline{f} - \bar{f}\|/2$. *A g in K is a best approximation if and only if* $\underline{f} - \Delta(f) \leqslant g \leqslant \bar{f} + \Delta(f)$. *Furthermore, if* $f' = (\bar{f} + \underline{f})/2$, *then* f' *is a best approximation and* $\|f' - h'\| \leqslant \|f - h\|$ *for all* $f, h \in X$. *The operator* $T\colon X \to K$ *defined by* $T(f) = f'$ *is an* OLSO *with* $c(T) = 1$.

THEOREM 2.2. *The following applies to Problem* (1.2) *for a nonconvex* $K \subset X$. *If K satisfies conditions* (1) *and* (2), *then* $\bar{f}$ *is the maximal best approximation to f and* $\bar{\Delta}(f) = \|f - \bar{f}\| = 2\Delta(f)$. *The operator* $T\colon X \to K$ *defined by* $T(f) = \bar{f}$ *is the unique* OLSO *with* $c(T) = 1$.

Proofs. The proofs presented are for the above theorems or Theorems 3.1 and 3.2 of [11] under conditions (1), (2), and (3).

We first prove Proposition 2.2 of [11]. If f, $h \in X$ and $\delta = \|f - h\|$ then $f - \delta \leqslant h$. Hence $\bar{f} - \delta \leqslant h$, and $\bar{f} - \delta \leqslant \bar{h}$, as $\bar{f} - \delta \in K$ by condition (1). By symmetry, $\|\bar{f} - \bar{h}\| \leqslant \|f - h\|$. The rest of the proof is as in Section 3 of [11]. Clearly, $\Delta = \Delta(f) = \|\underline{f} - \bar{f}\|/2$ in (c) follows from $\underline{f} \geqslant f \geqslant \bar{f}$, and $\underline{f} - \Delta \leqslant \bar{f} + \Delta$ and $2\Delta = \|\underline{f} - \bar{f}\|$. The exact values of $c(T)$ are obtained by examples as in [11] or [9, p. 212]. The outline of the proof is complete.

THEOREM 2.3 *Suppose that condition* (1) *holds for K.*

(a) (*For Problem* (1.1)). *Assume that the pointwise maximum* (*minimum*) *of two functions in K is also in K. If the maximal* (*minimal*) *best approximation to f exists, then condition* (2) ((3)) *holds. In this case, the maximal* (*minimal*) *best approximation equals* $\bar{f} + \Delta(f)$ ($\underline{f} - \Delta(f)$).

(b) (*For Problem* (1.2)). *Assume that the pointwise maximum of two functions in K is also in K. If the maximal best approximation to f exists, then condition* (2) *holds. In this case, the maximal best approximation equals* $\bar{f}$.

Proof. We establish (a); the proof for (b) is similar. Suppose that f' is the maximal best approximation. Then $f - \Delta(f) \leqslant f' \leqslant f + \Delta(f)$. If $f_0 = f' - \Delta(f)$, then $f_0 \leqslant f$, and hence, by condition (1), $f_0 \in K_f$. We show that $f_0 = \bar{f}$. Suppose not; then there exists k in K_f such that for some s in S we have $k(s) > f_0(s)$. Let $g = \max\{k, f_0\} + \Delta(f)$. By the hypothesis of the theorem we have $g \in K$. Clearly, $f - \Delta(f) \leqslant g \leqslant f + \Delta(f)$. Consequently, g is a best approximation and $g(s) > f'(s)$. Hence f' is not maximal, a contradiction. Thus, for all k in K_f, $k \leqslant f_0$ holds. Hence $\bar{f} = f_0$ and condition (2) holds. Also, $f' = \bar{f} + \Delta(f)$. The proof of the miminal case is similar. The proof is complete.

A unique best approximation is both the maximal and minimal best approximation. Hence $\Delta(f) = \|\underline{f} - \bar{f}\|/2$ of Theorem 2.1(c) and Theorem 2.3(a) yield the following for Problem (1.1): Suppose condition (1) holds and the pointwise maximum and minimum of two functions in K is also in K. Then a best approximation g is unique if and only if $\underline{f} - \bar{f} = \delta$ for some $\delta \geqslant 0$. In this case $\delta = 2\Delta(f)$ and $g = \underline{f} - \Delta(f) = \bar{f} + \Delta(f)$.

Let Z be the vector space of all operators with domain X and range in X. Let $Y \subset Z$ be the set of all selection operators T with domain X and range K for Problem (1.1). Clearly, Y is not necessarily convex. It is convex if K is convex. Furthermore, the set of all LSO's T is convex if K is convex and in this case $c(T)$ is a convex function of T [9]. T in Y is defined to be an extreme point of Y if $T = \lambda T_1 + (1 - \lambda) T_2$, $0 < \lambda < 1$, for some T_1 and T_2 in Y implies that $T_1 = T_2 = T$ [5].

PROPOSITION 2.1. *The* LSO T *defined in Theorem* 2.1(a) (2.1(b)), *which maps f to its maximal* (*minimal*) *best approximation, is an extreme point of* Y.

Proof. We prove the result for Theorem 2.1(a). Let $T = \lambda T_1 + (1-\lambda) T_2$, $0 < \lambda < 1$, where T_1 and T_2 are in Y. Since $T(f)$ is the maximal best approximation to f, we have $T_i(f) \leqslant T(f)$, $i = 1, 2$. Since $T = \lambda T_1 + (1-\lambda) T_2$, we have

$$\lambda(T(f) - T_1(f)) + (1-\lambda)(T(f) - T_2(f)) = 0,$$

which gives $T_i(f) = T(f)$, $i = 1, 2$. The proof is complete.

We remark that a similar result holds for the LSO T defined in Theorem 2.2. It would be interesting to know the nature and properties of extreme points of all LSO's.

3. DISTANCE FUNCTIONS AND NONCONVEX SETS IN R^n

The results of this section pertain to analysis of a function specifying the distance of a point from the complement of a convex set.

We denote the Euclidean norm of s in R^n by $|s|$. Let $\bar{A}$ be the closure of $A \subset R^n$. Let also $D(s, r)$ and $\bar{D}(s, r)$ denote, respectively, the open and closed balls with center s and radius r in R^n. For $A \subset R^n$, define the distance function $d(s, A)$ for s in R^n by

$$d(s, A) = \inf\{|s - t| : t \in A\},$$

[6, 7, 12]. It is known that d is Lipschitzian, i.e.,

$$|d(s, A) - d(t, A)| \leqslant |s - t|. \tag{3.1}$$

It can be easily shown that there exists t in $\bar{A}$ such that $d(s, A) = |s - t|$. If A is convex, then so is $\bar{A}$ and such a t is unique [6], and d is a convex function of s [12]. Furthermore, in this case, if t and v in $\bar{A}$ are nearest to s and u, respectively, then $|t - v| \leqslant |s - u|$, i.e., the mapping $s \to t$ is Lipschitzian. We examine the properties of d when A is not necessarily convex and has a special structure.

We briefly describe some terminology used in the sequel. Let $S \subset R^n$ be convex. Then aff(S), called the affine hull of S, is the smallest affine set containing S. Clearly $\bar{S} \subset \text{aff}(S)$. Let T denote aff(S). If $P \subset S \subset R^n$, where P and S are convex, then we denote by int(P) the interior of P when regarded as a subset of T with its relative topology. That is, $s \in \text{int}(P)$ if and only if there exists some $r > 0$ such that $D(s, r) \cap T \subset P$.

LEMMA 3.1. *Let* $P \subset S \subset R^n$, *where P and S are convex and S is closed. Let* $s \in \operatorname{int}(P)$ *and* $t \in \bar{P} \subset S$. *Then* $\lambda s + (1-\lambda) t \in \operatorname{int}(P)$ *for all* $0 < \lambda \leqslant 1$. *Also,* $\bar{P}$ *and* $\operatorname{int}(P)$ *are convex. Furthermore,* $\operatorname{aff}(P) = \operatorname{aff}(S)$ *if and only if* $\operatorname{int}(P) \neq \phi$. *If* $\operatorname{int}(P) \neq \phi$ *then* $\overline{\operatorname{int}(P)} = \bar{P}$.

Proof. The proof is similar to that of Theorems 6.1 and 6.2 of [8] and hence not given here. Note that the concepts of the relative interior ri C and of int C of a convex $C \subset R^n$ used in that proof are different from $\operatorname{int}(P)$ the concept we have defined. However ri $P = \operatorname{int}(P)$ if $\operatorname{int}(P) \neq \phi$.

PROPOSITION 3.1. *Let* $P \subset \operatorname{int}(S) \subset R^n$, *where P and S are convex, S is closed,* $\operatorname{int}(P) \neq \phi$, *and* $P \neq S$. *Then* $d(s, S \setminus P)$, $s \in R^n$, *is a concave function of s on* $\bar{P} \subset S$. (*It is Lipschitz continuous on* R^n *by* (3.1)).

Proof. For simplicitly denote $d(s, S \setminus P)$ by $d(s)$. Note that $d(s) > 0$ for s in S if and only if $s \in \operatorname{int}(P)$. Now let $s, t \in \bar{P}$, $0 < \lambda < 1$, and $u = \lambda s + (1-\lambda) t$. Suppose that $d(s) > 0$ and $d(t) > 0$. By definition of d, we conclude that the sets $D(s, d(s)) \cap S$ and $D(t, d(t)) \cap S$ are contained in $\operatorname{int}(P) \subset P$. Let E be the convex hull of these two sets. Then since $\operatorname{int}(P)$ is convex, we have $E \subset \operatorname{int}(P)$. It is easy to verify that $D(u, \lambda d(s) + (1-\lambda) d(t)) \cap S \subset E$. It follows that $d(u) \geqslant \lambda d(s) + (1-\lambda) d(t)$, which is the inequality for concavity of d. If $d(s) = d(t) = 0$, then this inequality holds. Now suppose that $d(s) > 0$ and $d(t) = 0$. Then $s \in \operatorname{int}(P)$ and $t \in \bar{P} \setminus \operatorname{int}(P)$. Define

$$F = \{v = \mu x + (1-\mu) t \colon x \in D(s, d(s)), 0 < \mu \leqslant 1\}.$$

Then F is convex (in fact a cone with apex t) and $t \in \bar{P} \setminus F$. By Lemma 3.1 we have $F \subset \operatorname{int}(P)$. Then as before we have $D(u, \lambda d(s)) \cap S \subset F$ and $d(u) \geqslant \lambda d(s)$. Thus d is concave on $\bar{P}$. We note that the concavity of d on $\bar{P}$ also follows from its concavity on $\operatorname{int}(P)$, continuity on $\bar{P}$, and a simple limit argument using Lemma 3.1. The proof is complete.

To prove the next proposition, we note that if $d(s)$ is a concave function on a convex set $A \subset R^n$, then the sets $\{s \in A \colon d(s) > r\}$ and $\{s \in A \colon d(s) \geqslant r\}$ are convex for all r [7].

PROPOSITION 3.2. *Assume that the hypothesis of Proposition* 3.1 *holds. For* $r > 0$, *sufficiently small, define*

$$Q = \{s \in S \colon d(s, S \setminus P) > r\}. \tag{3.2}$$

Then Q is nonempty convex, $\operatorname{int}(Q) = Q$, *and* $\bar{Q} \subset \operatorname{int}(P)$.

Proof. There exist x in P and $a > 0$ such that $d(x, a) \cap S \subset P$. For

$0 < r < a$, Q defined by (3.2) is not empty. For convenience, let $E = S \backslash P$. It is easy to see that

$$\text{int}(P) = \{s \in S: d(s, E) > 0\}$$

and, by convexity,

$$\bar{Q} = \{s \in S: d(s, E) \geqslant r\}.$$

Thus, $Q \subset \bar{Q} \subset \text{int}(P)$. By Proposition 3.1 we have that $d(s, E)$ is concave on $\bar{P}$ and also continuous. By (3.2), we conclude that Q is convex and $\text{int}(Q) = Q$. The proof is complete.

We remark that $Q = S \backslash F$, where

$$F = \bigcup \{\bar{D}(s, r): s \in E\}.$$

F and Q may be called parallel sets [11]. The results of this section, which are also of independent interest, may be used to obtain certain results in Section 4. However, the latter have been established by different methods. The results of this section will be referenced in other works.

4. Approximation Problems

We present examples of four approximation problems on C and one on B. We recall from convexity theory that a set $K \subset X$ is a cone if $\lambda k \in K$ whenever $k \in K$ and $\lambda \geqslant 0$. A cone K is a convex cone if $k + h \in K$ whenever $k, h \in K$. For convenience, we denote the set $\{s \in S: k(s) > \alpha\}$ by $\{k > \alpha\}$. Similar notation will be used for other sets.

Example 4.1. Approximation by convex functions in C.

Let $S \subset R^n$ be a polytope. A polytope is defined to be the convex hull of finitely many points [8]. It is compact and convex by definition. It is also locally simplicial [8]. We use this property to derive our results. Let $C = C(S)$ and K be the set of all convex functions in C. It is easy to verify that K is a closed convex cone. Clearly, condition (1) holds for K. We consider problems (1.1) and (1.2) in this setting. The greatest K-minorant of f is called the greatest convex minorant of f.

Proposition 4.1. *If $f \in C$ then $\bar{f}$ is convex and continuous. Hence $\bar{f}$ is the greatest convex minorant of f.*

Proof. Since $\bar{f}$ is the pointwise supremum of convex functions, it is convex. Since S is locally simplicial, continuity of $\bar{f}$ on S follows from Corollary 17.2.1 and Theorem 10.2 of [8]. The proof is complete.

The proposition shows that condition (2) also holds for K. Hence Theorems 2.1(a) and 2.2 apply. The operator T defined in Theorem 2.1(a) is an LSO with $c(T)=2$. The example given in [9, p. 212] on $S=[0, 1]$ with f_n and f continuous shows that $c(T')\geqslant 2$ for all LSO T'. Thus T is an OLSO with $c(T)=2$.

EXAMPLE 4.2. Approximation by quasi-convex functions in C.

Let $S \subset R^n$ be convex and compact, and $C = C(S)$. A function k in $B=B(S)$ is called quasi-convex if

$$k(\lambda s+(1-\lambda)\, t)\leqslant \max\{k(s), k(t)\}$$

for all s, t in S, all $0\leqslant\lambda\leqslant 1$ [7]. Equivalently, k in B is quasi-convex if one of the following conditions holds: (i) $\{k\leqslant\alpha\}$ is convex for all real α, (ii) $\{k<\alpha\}$ is convex for all real α. Let K be the set of all quasi-convex functions in B. It is easy to verify that K is a closed cone but it is not convex. Clearly, K satisfies condition (1). We consider Problems (1.1) and (1.2) in this setting. The greatest K-minorant of f is called the greatest quasi-convex minorant of f.

Let Π be the set of all convex subsets of S. Since S, $\phi\in\Pi$, it is not empty. For $A\subset R^n$, we denote by $\mathrm{co}(A)$ the convex hull of A, i.e., the smallest convex set containing A.

PROPOSITION 4.2. *Let $f\in C$ and define*

$$f^0(P)=\inf\{f(t)\colon t\in S\setminus P\}, \qquad P\in\Pi,$$

$$\bar f(s)=\sup\{f^0(P)\colon P\in\Pi, s\in S\setminus P\}, \qquad s\in S.$$

Then $\bar f$ is quasi-convex and continuous. It is the greatest quasi-convex minorant of f. Furthermore, an h in B is the greatest quasi-convex minorant of f if and only if, for all real α,

$$\{h<\alpha\}=\mathrm{co}\{f<\alpha\}, \tag{4.1}$$

or, equivalently, for all real α,

$$\{h\leqslant\alpha\}=\mathrm{co}\{f\leqslant\alpha\}. \tag{4.2}$$

Proof. We first show that $\bar f$ is continuous by assuming that $\bar f$ is the greatest quasi-convex minorant, and (4.1) and (4.2) apply to $\bar f$. We will prove these assertions independently later. Since $f\in C$, $\{f\leqslant\alpha\}$ is compact. Hence, by (4.2), its convex hull $\{\bar f\leqslant\alpha\}$ is compact. Thus $\{\bar f>\alpha\}$ is open in S. Again, $\{f<\alpha\}$ is open in S and by a result similar to [7, p. 78, Theorem G] for relative topologies and by (4.1), its convex hull $\{\bar f<\alpha\}$ is open in S. Thus $\bar f$ is continuous.

The equivalence of (4.1) and (4.2) is established below. The rest of the proof including that of the validity of (4.1) is a minor modification of the proof of Theorem 3.1 of [10]. However, for the convenience of the reader we show that $\bar{f}$ is the greatest quasi-convex minorant of f.

Let $s, t \in S$, $0 < \lambda < 1$, and $u = \lambda s + (1-\lambda)t$. Given $\varepsilon > 0$, there exists P in Π with $u \in S \backslash P$ such that $\bar{f}(u) \leqslant f^0(P) + \varepsilon$. Since P is convex, we conclude that $s \in S \backslash P$ or $y \in S \backslash P$. Hence $f^0(P) \leqslant \max\{\bar{f}(s), \bar{f}(t)\}$ and $\bar{f} \in K$. Clearly, $f^0(P) \leqslant f(s)$ for all P in Π with s in $S \backslash P$ and, hence, $\bar{f} \leqslant f$. Thus $\bar{k} \leqslant k$ if $k \in K$. Now let $k \in K$, $s \in S$, and $Q = \{t \in S: k(t) < k(s)\}$. Then $Q \in \Pi$ and $s \in S \backslash Q$. Hence $k^0(Q) \geqslant k(s)$ and $\bar{k} \geqslant k$. Hence $\bar{k} = k$. If $k \in K$ and $k \leqslant f$, then we conclude that $k^0(P) \leqslant f^0(P)$ for all P in Π, which gives $k = \bar{k} \leqslant \bar{f}$. Thus $\bar{f}$ is the greatest quasi-convex minorant of f.

To show the equivalence of (4.1) and (4.2), suppose that (4.1) holds for all α. We show (4.2). Then

$$\{h \leqslant \alpha\} = \bigcap_m \{h < \alpha + 1/m\} = \bigcap_m \operatorname{co}\{f < \alpha + 1/m\},$$

where m denotes a positive integer. Hence it suffices to show that

$$\operatorname{co}\{f \leqslant \alpha\} = \bigcap_m \operatorname{co}\{f < \alpha + 1/m\}. \tag{4.3}$$

Let $A_m = \{f < \alpha + 1/m\}$. Since $\{f \leqslant \alpha\} \subset A_m$ for each m, we conclude that the left-hand side of (4.3) is contained in the right-hand side. To show the reverse inclusion, let $s \in \bigcap_m \operatorname{co}(A_m)$. Then $s \in \operatorname{co}(A_m)$ and, by Caratheodory's theorem [7, 8], there exist points $s_{i,m}$, $0 \leqslant i \leqslant n$, in A_m such that $s \in \operatorname{co}\{s_{i,m}\}$. Hence, $s = \sum_i \lambda_{i,m} s_{i,m}$, where $\lambda_{i,m} \geqslant 0$ and $\sum_i \lambda_{i,m} = 1$. By compactness of $[0, 1]$ and S, some subsequences of $(\lambda_{i,m})$ and $(s_{i,m})$ converge for each i, as $m \to \infty$, to some λ_i and s_i with $\sum \lambda_i = 1$ and $s = \sum \lambda_i s_i$. By continuity of f we have $f(s_i) \leqslant \alpha$ for all i. Thus $s \in \operatorname{co}\{f \leqslant \alpha\}$. Hence (4.1) implies (4.2). Now suppose that (4.2) holds for all α. Then

$$\{h < \alpha\} = \bigcup_m \{h \leqslant \alpha - 1/m\} = \bigcup_m \operatorname{co}\{f \leqslant \alpha - 1/m\}.$$

Hence, to show (4.1), it suffices to show that

$$\operatorname{co}\{f < \alpha\} = \bigcup_m \operatorname{co}\{f \leqslant \alpha - 1/m\}. \tag{4.4}$$

Clearly, the right-hand side of (4.4) is contained in the left-hand side. Hence, let $t \in \operatorname{co}\{f < \alpha\}$. Then there exist t_i, $0 \leqslant i \leqslant n$, with $f(t_i) < \alpha$ such that $s \in \operatorname{co}\{t_i\}$. Hence $f(t_i) \leqslant \alpha - 1/m_0$ for some m_0 and all i. Thus $s \in \operatorname{co}\{f \leqslant \alpha - 1/m_0\}$ and (4.2) implies (4.1). The proof is complete.

We conclude that condition (2) holds for $K \cap C$. Hence, Theorems 2.1(a) and 2.2 apply to $X = C$ and $K \cap C$. The operator $T: C \to K \cap C$ defined by Theorem 2.1(a) is an LSO with $c(T) = 2$. The example given in [10] on $S = [0, 3]$ with f_n and f continuous shows that $c(T') \geq 2$ for all LSO T'. Hence T is an OLSO with $c(T) = 2$.

EXAMPLE 4.3. Approximation by isotone functions and its variants on B and C.

We first introduce this problem on B. Let S be any set. Let Π be the set of subsets of S such that ϕ, $S \in \Pi$ and Π is closed under arbitrary unions. Motivation for this definition of Π comes from the special case of partially ordered sets given later in this section. A function k in $B = B(S)$ is called Π-isotone if $\{k > \alpha\} \in \Pi$ for all real α. Let $K \subset B$ be the set of all Π-isotone functions. It is easy to verify that K is a cone. We show that it is closed. Suppose (k_n) is a sequence in K such that $\|k_n - k\| = \delta_n \to 0$ for some k in B. Then

$$\{k > \alpha\} = \bigcup_n \{k_n > \alpha + \delta_n\},$$

which shows that $k \in K$ and K is closed. Clearly, condition (1) holds for K. We consider problems (1.1) and (1.2) in this setting. Note that Example 4.2 may be transformed to this setting.

PROPOSITION 4.3. *Let $f \in B$ and define*

$$f^0(P) = \inf\{f(t): t \in P\}, \qquad P \in \Pi, \tag{4.5}$$

$$\bar{f}(s) = \sup\{f^0(P): P \in \Pi, s \in P\}, \qquad s \in S. \tag{4.6}$$

Then $\bar{f} \in K$ and is the greatest K-minorant of f.

Proof. We first show that $\bar{f} \in K$. Let $A = \{\bar{f} > \alpha\}$. If $s \in A$, then there exists $P_s \in \Pi$ such that $s \in P_s$ and $f^0(P_s) > \alpha$. Now if $u \in P_s$, then clearly $\bar{f}(u) \geq f^0(P_s) > \alpha$. Hence $A \supset P_s$ and $A = \bigcup\{P_s: s \in A\}$ is in Π. Thus $\bar{f} \in K$. Clearly, $f^0(P) \leq f(s)$ for all P in Π with s in P. Hence $\bar{f} \leq f$. In particular, $\bar{k} \leq k$ if $k \in K$. Suppose now that $k \in K$. Let $\varepsilon > 0$, $s \in S$, and $P = \{t \in S: k(t) > k(s) - \varepsilon\}$. Then $P \in \Pi$ and $s \in P$. Hence $\bar{k}(s) \geq k^0(P) \geq k(s) - \varepsilon$ or $\bar{k} \geq k$. Thus $\bar{k} = k$. If $k \in K$ and $k \leq f$, then clearly $k = \bar{k} \leq \bar{f}$. Thus $\bar{f}$ is the greatest K-minorant of f. The proof is complete.

Thus condition (2) also holds for K and Theorems 2.1(a) and (2.2) apply.

Now suppose that Π is closed under both arbitrary unions and intersections. Let K be defined as before. Then K is a closed convex cone. Note that the convexity of K follows from the equality,

$$\{k+h>\alpha\}=\bigcup_{\delta}\{\{k>\delta\}\cap\{h>\alpha-\delta\}\}.$$

Closure of Π under both unions and intersections implies that $k\in K$ if and only if $\{k\geqslant\alpha\}\in\Pi$ for all α. For $s\in S$ let U_s be the intersection of all sets P in Π such that $s\in P$. Then $U_s\in\Pi$ and is the smallest set in Π containing s. Similarly, let V_s be the union of all sets P in Π such that $s\in S\backslash P$. Then $V_s\in\Pi$ and is the largest set in Π such that $s\in S\backslash V_s$.

PROPOSITION 4.4 *Let $f\in B$ and define*

$$\bar{f}(s)=\inf\{f(t)\colon t\in U_s\},\qquad s\in S,\tag{4.7}$$

$$\underline{f}(s)=\sup\{f(t)\colon t\in S\backslash V_s\},\qquad s\in S.\tag{4.8}$$

Then $\bar{f}$ and $\underline{f}\in K$ and are, respectively, the greatest K-minorant and the smallest K-majorant of f.

Proof. Clearly, (4.5) and (4.6) give (4.7) under the stronger conditions of the proposition. To show the assertion concerning $\underline{f}$, define $\Pi'=\{S\backslash P\colon P\in\Pi\}$. Then Π' is closed under arbitrary unions and intersections. Clearly, $k\in K$ if and only if $\{k\geqslant\alpha\}\in\Pi$ for all α, which is equivalent to $\{-k>\alpha\}\in\Pi'$ for all α. Thus $-K$ equals all k in B such that $\{k>\alpha\}\in\Pi'$ for all α. Then, by the proof for $\bar{f}$ as applied to $-K$ and Π', we conclude that the greatest $-K$-minorant of $-f$ exists in $-K$. Substituting $-f$ for f and $S\backslash V_s$ for U_s in (4.7) we verify that it is given by $-\underline{f}$. Then $\underline{f}$ in K is the smallest K-majorant of f. The proof is complete.

Thus conditions (2) and (3) hold for K and Theorems 2.1(c) and 2.2 apply.

We now specialize the results to a partially ordered set. A partial order $\leqslant$ on S is a reflexive, transitive, but not necessarily an antisymmetric relation [5]. A subset U of a partially ordered set S is called an upper set if $s\in U$, $t\in S$, and $s\leqslant t$ implies that $t\in U$. Similarly, $L\subset S$ is a lower set if $s\in L$, $t\in S$, and $t\leqslant s$ implies that $t\in L$. Clearly ϕ, S are both upper and lower sets and U is an upper set if and only if $L=S\backslash U$ is a lower set. A function k in B is said to be isotone (order preserving) if $k(s)\leqslant k(t)$ whenever $s\leqslant t$. Let Π_0 be the set of all upper sets of S. Then, clearly Π_0 is closed under unions and intersections. The following lemma is immediate.

LEMMA 4.1. *k on S is isotone if and only if it is Π_0-isotone.*

For this case Proposition 4.4 applies. We obtain $\bar{f}$ and $\underline{f}$ by (4.7) and (4.8), where U_s and $L_s=S\backslash V_s$ are, respectively, the smallest upper and lower sets containing s. Clearly $U_s=\{t\colon s\geqslant t\}$ and $L_s=\{t\colon t\leqslant s\}$.

Theorems 2.1(c) and 2.2 then hold. See [9] for such a problem with additional constraints.

Now we consider an application to C. Let $S = \times\{[a_i, b_i]: 1 \leqslant i \leqslant n\} \subset R^n$, where $a_i < b_i$, and $\leqslant$ be the usual partial order on vectors. We let $C = C(S)$ and K be the set of all isotone functions in C. It is easy to verify that K is a closed convex cone.

PROPOSITION 4.5. *Let $f \in C$ and define*

$$\bar{f}(s) = \min\{f(t): t \in S, t \geqslant s\},$$

$$\underline{f}(s) = \max\{f(t): t \in S, t \leqslant s\}.$$

Then $\bar{f}$ and $\underline{f} \in K$ and are, respectively, the greatest K-minorant and smallest K-majorant of f.

Proof. The expressions for $\bar{f}$ and $\underline{f}$ follow from the above discussion. Clearly, $\bar{f}$ and $\underline{f}$ are isotone. We show that $\bar{f}$ is continuous. For simplicity, we prove this when $S \subset R^2$; the proof for the general case is similar. Let $s = (s_1 . s_2)$ denote an element of R^2. By uniform continuity of f, given $\varepsilon > 0$, there exists $\delta > 0$ such that if $s, t \in S$ and $|s - t| < \delta$ then $|f(s) - f(t)| < \varepsilon$. For such s, t in S, suppose first that $s \nleqslant t$. Without loss of generality, assume $t_1 < s_1$ and $t_2 > s_2$. Let $v = (s_1, t_2)$. Define $U_x = \{u \in S: u \geqslant x\}$ for $x = s, t, v$. Then, by the definition of $\bar{f}$ and $\underline{f}$, we have

$$\bar{f}(s) = \min\{\inf\{f(u): u \in U_s \setminus U_v\}, \bar{f}(v)\},$$

$$\bar{f}(t) = \min\{\inf\{f(u): u \in U_t \setminus U_v\}, \bar{f}(v)\}.$$

Let $u \in U_t - U_v$ and $w = (s_1, u_2)$. Then $w \in U_v$ and $|w - u| < \delta$. Hence $f(u) \geqslant f(w) - \varepsilon$ and

$$\inf\{f(u): u \in U_t - U_v\} \geqslant \inf\{f(u): u \in U_v\} - \varepsilon = \bar{f}(v) - \varepsilon.$$

Then, we have $\bar{f}(t) \geqslant \bar{f}(v) - \varepsilon$ and also $\bar{f}(s) \leqslant \bar{f}(v)$ since $s \leqslant v$. Hence $\bar{f}(s) - \bar{f}(t) \leqslant \varepsilon$. In a symmetric manner, we have $\bar{f}(t) - \bar{f}(s) \leqslant \varepsilon$. Now if $s \leqslant t$ then we may similarly show that $|\bar{f}(s) - \bar{f}(t)| \leqslant \varepsilon$. Hence $\bar{f}$ is continuous and is in K. Similarly, $\underline{f}$ is continuous and is in K. Now if k, $h \in K$ and $k \leqslant f \leqslant h$, then clearly $k \leqslant \bar{f} \leqslant \underline{f} \leqslant h$. Thus the last assertion of the proposition is established. The proof is complete.

We conclude that Theorems 2.1(c) and 2.2 apply.

EXAMPLE 4.4. Approximation by Lipschitzian functions in C.

Let S be a real compact interval and $C = C(S)$. Let K be all functions k in C satisfying, for some fixed M and α [1], the condition

$$|k(s) - k(t)| \leqslant M\,|s - t|^{\alpha}. \tag{4.9}$$

It is easy to verify that K is closed and convex. We consider problems (1.1) and (1.2) in this setting. Clearly, condition (1) holds for K.

It is easy to show that both $\bar{f}$ and $\underline{f}$ satisfy (4.9) and, hence, are in K. Thus conditions (2) and (3) hold for K. Then Theorems 2.1(c) and 2.2 apply.

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